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Transport by Vector Fields with Kolmogorov Spectrum

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We study a model of turbulent transport described by the motion in a Gaussian random velocity field with Kolmogorov spectrum. The field is assumed to be divergence-free, homogeneous in time and space, and Markovian in time. The molecular viscosity defines the cutoff in the Fourier space, thus regularizing the vector field of the pure infinite-Reynolds-number Kolmogorov spectrum by vector fields with smooth realizations. We provide an asymptotic bound on the effective diffusivity of the finite-Reynolds number fields as $R \to \infty$. Namely, with macroscopic parameters of the system fixed and the viscosity tending to zero, the effective diffusivity is bounded above by a constant which does not depend on the Reynolds number.

KEY WORDS: Turbulence; random fields; Kolmogorov spectrum; effective diffusivity.

1. INTRODUCTION

Consider the motion of a particle in the random velocity field $V(x, t)$, $x \in \mathbb{R}^3$, which is described by the system of random ordinary differential equations

$$\dot{X}_t = V(X_t, t), \qquad X_0 = x_0 \tag{1}$$

The initial data x_0 is assumed to be independent of the velocity field $V(x, t)$. The matrix of effective diffusivity is defined as

$$D^{ab} = \frac{1}{2} \lim_{t \to \infty} \frac{E((X_t^a - X_0^a)(X_t^b - X_0^b))}{t}, \qquad a, b = 1,..., 3 \tag{2}$$

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where a and b are coordinate directions. It needs to be stressed that we are considering the long-time limit, when fluid particle displacements obey $|X_t - X_0| \gg L$, where L is the characteristic length scale of the flow.

In a series of papers[8, 9] we have proved the existence of effective diffusivity for Markovian in time Gaussian flows with smooth realizations. See also related work of Fannjiang, Komorowski, and Papanicolaou.[2, 7] Here we relate our results to the Kolmogorov theory. Our main result is a rigorous proof of the upper bound for the effective diffusivity on the basis of Kolmogorov structure assumed for V.

We start with a vector field whose spectral measure is of order $|k|^\alpha$ at infinity. The case of pure Kolmogorov spectrum corresponds to $\alpha = -\frac{11}{3}$. In this case a typical realization of the field V is not Lipschitz continuous.

In order to make sense of the equation of motion (1) we introduce cutoffs at infinity for the spectral measure, thus approximating V by smooth random fields V^m. The spectral measure of V^m is defined to be equal to the spectral measure of V on the cube of size m centered at the origin in wave number space, and equal to zero elsewhere. As follows from ref. 10 the size of the cutoff m scales as $R^{3/4}/L$, where R is the Reynolds number and L is the macroscopic length scale of the flow.

In Section 2, following ref. 11, and also using ideas which succeeded in constructive quantum field theory,[5] we use a discretization of the spectrum of the random field $V^m(x, t)$ in order to approximate the system of random ordinary differential equations $\dot{X}_t = V^m(X_t, t)$ by a finite dimensional system of stochastic differential equations. The resulting vector field, whose spectrum is supported on a finite number of points, will be denoted by V^{mn}. Thus, together with (1) we consider an auxiliary system

$$\dot{X}_t = V^{mn}(X_t, t) \tag{3}$$

Here m is the size of the cutoff and n is the number of points in the support of the spectrum of V^{mn}. The same discretization procedure was used relative to the cutoff n in ref. 9. Now we additionally need to keep track of the dependence of V^{mn} on the large wavenumber cutoff m.

From the Markov assumption governing the time correlations of the random velocity statistics, each Fourier mode in the random velocity field $V^{mn}(x, t)$ is represented by a multidimensional Ornstein–Uhlenbeck process. Thus (3) can be also viewed as a system of stochastic differential equations

$$dX^a_t = \sum_{i=1}^{2n} Y^i_t v^a_i(X_t)\, dt, \qquad a = 1,..., 3 \tag{4}$$

$$dY^i_t = \sqrt{2\Omega_i}\, dW^i_t - \Omega_i Y^i_t\, dt, \qquad i = 1,..., 2n \tag{5}$$

where v_i are periodic with common period p, and Y^i_t are independent Ornstein–Uhlenbeck processes. The process Y^i corresponds to the Fourier mode with wavenumber k_i of the field V^{mn}. The speed of the process, Ω_i, is equal to the decay rate of the time correlation of the mode with frequency k_i. The Kolmogorov assumption implies that the dependence of Ω on k is

$$\Omega(k) \sim |k|^{2/3} \tag{6}$$

for large $|k|$.[10]

In Section 3 we use the harmonic coordinates method[12, 4] to express the effective diffusivity of the field V^{mn} in terms of the solutions u^a of the hypoelliptic equations

$$M\left(\frac{u^a}{\eta} + x_a\right) = 0$$

on $\mathbb{T}^3_p \times \mathbb{R}^{2n}$, where M is the infinitesimal generator of the system (4), (5), and η^2 is its invariant measure. The existence and uniqueness of solutions to this PDE is one of the main technical results of ref. 8. Hormander's hypoellipticity principle[6] applied to the differential operator and its adjoint is a key element in the proof of existence and regularity. We state the existence and uniqueness theorem in Section 3, referring the reader to ref. 8 for the proof.

In Section 4 we obtain an a priori estimate for the operator M which is uniform in the size of the cutoff m and in the number of modes n in the spectrum of the velocity field. The power-law growth (6) for the decay rates of the velocity modes is crucial for the proof of the a priori estimate. The estimate allows us to bound the effective diffusivity $D^{m,\,ab}$ of the fields V^m. We obtain

$$D^{m,\,ab} \leqslant c \tag{7}$$

for the vector fields with Kolmogorov spectrum, as a rigorous bound in agreement with formal perturbation theory.

2. DEFINITIONS, ASSUMPTIONS, AND RESULTS

Let $V(x, t)$, $x \in \mathbb{R}^d$ be a divergence free, zero mean Gaussian vector field, which is stationary in x and t and Markov in time. Let $G^{ab} = E(V^a(x, t)\, V^b(0, 0))$ be the correlation matrix of the field $V(x, t)$.

The properties of $V(x, t)$ listed above imply the exponential in time behavior for the spectral matrix $\widehat{G^{ab}}(k, t) = (2\pi)^{-3} \int e^{-ikx} G^{ab}(x, t)\, dx$ of V.[11]

Namely,

$$\hat{G}(k, t) = \exp(-|t|\ \Omega(k))\ M(k) \tag{8}$$

for some matrices Ω and M. We shall use (8) to define a class of vector fields, which will include the fields whose finite dimensional distributions are invariant with respect to orthogonal transformations of space variables (isotropy), and time reversal. Namely, we shall assume that $\Omega(k)$ is a scalar, and the matrix $M(k)$ is symmetric. In the isotropic case $M(k) = \phi(|k|)(\delta^{ab} - (k^a k^b/|k|^2))$.[1]

The condition $(M(k)\, k, k) = 0$ for all k is equivalent to the divergence free property of the field. Recall, that since M is a Fourier transform of a positive definite matrix valued function, it follows that for each a, b fixed, $M^{ab}(dk)$ is a real valued signed measure on $\mathbb{R}^3$, and that $M(k)$ is a positive matrix. The positivity means that for each vector $v \in \mathbb{R}^3$, and each measurable set $A \subset \mathbb{R}^3$

$$\sum_{a, b=1}^{3} v^a v^b \int_A M^{ab}(dk) \geqslant 0$$

We shall denote the variation of $M^{ab}(dk)$ by $|M^{ab}|\ (dk)$.

Let us discuss the physical interpretation of formula (8) in the case of the three dimensional turbulence with Kolmogorov spectrum. In the Kolmogorov picture turbulence is looked upon as a system of eddies corresponding to different velocity frequencies. The matrix $M(k)$ is of order of the density of the kinetic energy corresponding to modes with frequency k. From the dimensional considerations it follows that

$$M(k) \sim |k|^{-11/3} \tag{9}$$

for large $|k|$.[10] It is worth noting that the results apply also if we allow some intermittency correction to the spectrum $M(k) \sim |k|^{-11/3+\mu}$, with $0 < \mu < 4/3$, as in the Kolmogorov refined similarity theory.

With the characteristic length scale L, and the characteristic velocity U fixed, the size of the domain in which (9) is valid is determined by the fluid viscosity. Namely, outside of the cube of size m, the matrix $M(k)$ is assumed to be rapidly decreasing. The size of the cube, m, is proportional to the ratio of the 3/4 power of the Reynolds number R and the macroscopic length scale L of the flow, $m \sim R^{3/4}/L$.

Since we are interested in upper estimates on the effective diffusivity, we assume that $|M^{ab}(k)| \leqslant c(1+|k|)^{-11/3}$ for some c and all a, b, and k. We shall study the dependence of the effective diffusivity on the Reynolds number R, or more precisely, on the cutoff m in the velocity spectrum.

As seen from (8) the function $\Omega(k)$ is the decay rate, or inverse life time of an eddy with frequency k. As follows from ref. 10, classical Kolmogorov turbulence corresponds to the case $\Omega(k) \sim |k|^{2/3}$ for large $|k|$.

We now state the assumptions on the stream function, in somewhat more generality than needed for the case of pure Kolomogorov spectrum of the velocity field.

Assumption A. $V(x, t)$, $x \in \mathbb{R}^3$ is a zero mean Gaussian field, stationary in x and t, isotropic in x, and Markov in time.

Assumption B. The spectral matrix of the field V is given by (8), where $\Omega(k)$ is scalar, and the matrix $M(k)$ is symmetric. There exists a constant $c > 0$, and a compact set $K \subset \mathbb{R}^3$, such that

$$\int_K dM(k) > cI$$

where I is the identity matrix. There exist constants $c_1, c_2 > 0$, and α, β, such that

$$\alpha - \beta + 3 < 0$$

$$|M^{ab}|\,(k) \leqslant c_1(1 + |k|)^{\alpha} \tag{10}$$

$$\Omega(k) \geqslant c_2(1 + |k|)^{\beta} \tag{11}$$

Moreover, $\Omega(k)$ is Lipschitz continuous uniformly on any compact.

The case of pure Kolmogorov spectrum for three dimensional turbulence corresponds to $\alpha = -11/3$ and $\beta = 2/3$. The vector field $V^m(x, t)$ is defined to be the real valued Gaussian random field whose spectral matrix $\widehat{G^m}$ is given by (8) on the set $\{|k^a| < m, a = 1,..., 3\}$, and is equal to zero outside this set. Thus,

$$\widehat{G^m}(k, t) = \exp(-|t|\,\Omega(k))\,M^m(k)$$

where

$$M^m(k) = M(k)\,\chi_{\{|k^a| < m\}}$$

It was shown in ref. 9 that under the above assumptions there exists m_0 such that the effective diffusivity of the field V^m for $m \geqslant m_0$ exists and is finite. We shall denote it by $D^{m,\,ab}$. We now formulate the main theorem of this paper.

Theorem 2.1. Suppose Assumptions A and B hold. Then there exists a constant c such that $|D^{m, ab}| < c$, $a, b = 1,..., 3$ for all $m > m_0$.

We next describe the approximation of the vector field $V^m(x, t)$ by the vector fields $V^{mn}(x, t)$, whose spectral matrices $\widehat{G^{mn}}(k, t)$ are supported on finite sets in k-space.

Consider the partition of the cube $Q_m = \{|k^a| \leqslant m, a = 1,..., 3\}$ into $n = (2l)^3$ cubes Δ_i, $i = 1,..., n$ of the size m/l. Let k_i be the center of Δ_i. Let α_i be the interior of Δ_i, β_i be the boundary of Δ_i excluding the edges, γ_i be the edges without the endpoints, and δ_i be the set which consists of six vertices of the cube Δ_i. Define

$$\Omega_i = \Omega(k_i)$$

and

$$N_i = \int_{\alpha_i} M^m(dk) + \tfrac{1}{2} \int_{\beta_i} M^m(dk) + \tfrac{1}{4} \int_{\gamma_i} M^m(dk) + \tfrac{1}{6} \int_{\delta_i} M^m(dk) \tag{12}$$

It is important to note that Δ_i, k_i, Ω_i, α_i, β_i, γ_i, δ_i, and N_i depend on m and n. The first step in the transition from $M^m(k)$ to $M^{mn}(k)$ consists of integrating $M^m(k)$ over each cube, and placing all the mass in the center. The four different integrals enter (12) with their specified factors because each side belongs to two different cubes, each edge—to four different cubes, and each vertex to six different cubes. The second step is designed to make the matrices M^{mn} satisfy the condition $(M^{mn}(k)\, k, k) = 0$. We define P_i to be the orthogonal projection on the subspace orthogonal to k_i. Define

$$M_i^{mn} = P_i N_i P_i; \qquad M^{mn}(k) = \sum_{i=1}^{n} \delta(k_i)\, M_i^{mn}$$

and

$$\widehat{G^{mn}}(k, t) = \exp(-|t|\, \Omega(k))\, M^{mn}(k) \tag{13}$$

Then $V^{mn}(x, t)$ is defined to be the real valued Gaussian random field whose spectral matrix is given by (13).

The Fourier representation of the field $V^{mn}(x, t)$ in space variables is

$$V^{mn}(x, t) = \int e^{ikx} Z(dk, t)$$

For t fixed $Z(k, t)$ is an orthogonal Gaussian measure, which depends on m and n. Since supp $\widehat{V^{mn}}(k, 0) \subset \{k_i\}$

$$V^{mn}(x, t) = \sum_{i=1}^{n} e^{ik_i x} z(k_i, t) \tag{14}$$

where $z(k_i, t)$ are complex vector-valued Gaussian stationary processes. The normalization of $z(k_i, t)$ is fixed by (13) so that

$$E(z^a(k_i, t)\, \overline{z^b}(k_j, 0)) = \delta_{ij} M_i^{mn, ab} \exp(-|t|\, \Omega_i)$$

The fact that $V(x, t)$ is a real valued field implies that together with the mode k_i the set $\{k_i\}$ also contains $-k_i$ with the same M_i^{mn} and Ω_i. We shall write

$$\{k_i, i = 1,..., n\} = \{k_i, -k_i, i = 1,..., n/2\}$$

Therefore we can write (14) as

$$V^{mn}(x, t) = \sum_{i=1}^{n/2} (A_{i1}(t) \cos(k_i x) + A_{i2}(t) \sin(k_i x)) \tag{15}$$

Here $A_{il}(t)$, $i = 1,..., n/2$; $l = 1, 2$ are independent real vector-valued stationary Gaussian processes and

$$E(A_{il}^a(t)\, A_{i'l'}^b(0)) = 2\delta_{ii'}\, \delta_{ll'} M_i^{mn, ab} \exp(-|t|\, \Omega_i)$$

This implies that the $A_{il}(t)$ are independent vector valued Ornstein–Uhlenbeck processes with correlation scales Ω_i and variances $2M_i^{mn, ab}$.

Recall that M_i^{mn} is an orthogonal matrix, and k_i is its eigenvector with eigenvalue 0. Let e_i^1, e_i^2 and λ_i^1, λ_i^2 be the eigenvectors and eigenvalues of M_i^{mn} in the subspace orthogonal to k_i. Note that

$$A_{il} = \sqrt{2}\, (\sqrt{\lambda_i^1}\, e_i^1 B_{il}^1 + \sqrt{\lambda_i^2}\, e_i^2 B_{il}^2)$$

where B_{il}^1 and B_{il}^2 are independent Ornstein–Uhlenbeck scalar valued processes with correlation scales Ω_i and variances 1. We shall write

$$\{Y^i, i = 1,..., 2n\} = \{B_{il}^j, i = 1,..., n/2; l = 1, 2; j = 1, 2\}$$

We shall use the same notation Ω_i for the correlation scale of the process Y^i. Thus (15) becomes

$$V^{mn}(x, t) = \sum_{i=1}^{2n} Y^i(t)\, v_i(x) \tag{16}$$

where the vectors $v_i(x)$, $i=1,\dots, 2n$ are of the following form

$$\{v_i, i=1,\dots, 2n\} = \{\sqrt{2\lambda_i^1}\, e_i^1 \cos k_i x, \sqrt{2\lambda_i^1}\, e_i^1 \sin k_i x, \sqrt{2\lambda_i^2}\, e_i^2 \cos k_i x, \sqrt{2\lambda_i^2}\, e_i^2 \sin k_i x, i=1,\dots, n/2\} \tag{17}$$

Therefore the vector fields v_i are divergence free and infinitely smooth. By the definition of k_i the vector fields are periodic with common period $p=4l\pi/m$.

The effective diffusivity of the vector field V^{mn} will be denoted by $D^{mn,ab}$. It was shown in ref. 9 that under Assumptions A and B there exists m_0, such that $D^{mn,ab} \to D^{m,ab}$ as $n \to \infty$ for $m \geqslant m_0$. Therefore Theorem 2.1 is a consequence of the following

Theorem 2.2. Suppose Assumptions A and B hold. Then there exists a constant c such that $|D^{mn,ab}| < c$, $a, b=1,\dots, 3$ for $m \geqslant m_0$, $n \geqslant n_0(m)$.

3. THE HYPOELLIPTIC EQUATION

In this section we express the effective diffusivity $D^{mn,ab}$ in terms of the solution of a hypoelliptic PDE. The estimate of the solution which leads to the desired estimate of $D^{mn,ab}$ is proved in Section 4.

By (16) the equation of motion in the vector field $V^{mn}(x,t)$ has the form

$$dX_t^a = \sum_{i=1}^{2n} Y_t^i v_i^a(X_t)\, dt, \qquad a=1,\dots, 3 \tag{18}$$

where the Y_t^i are independent Ornstein–Uhlenbeck processes

$$dY_t^i = \sqrt{2\Omega_i}\, dW_t^i - \Omega_i Y_t^i\, dt, \qquad i=1,\dots, 2n \tag{19}$$

The operator

$$M = \sum_{i=1}^{2n} \Omega_i(\partial_{y_i}^2 - y_i \partial_{y_i}) + \sum_{i=1}^{2n} y_i v_i(x) \nabla_x \tag{20}$$

is the infinitesimal generator of the system (18)–(19). Recall that p is the common period of the velocity modes defined in Section 2. Let

$$\eta(x, y) = p^{-3/2} \prod_{i=1}^{2n} (2\pi)^{-1/4} \exp\left(-\frac{y_i^2}{4}\right)$$

As initial conditions for the system we take the distribution η^2, which is invariant for the process (X_t, Y_t) on $\mathbb{T}_p^3 \times \mathbb{R}^{2n}$.

Consider the equation

$$M\left(\frac{u^a}{\eta} + x_a\right) = 0 \tag{21}$$

for a function $u^a(x, y)$ defined on $\mathbb{T}_p^3 \times \mathbb{R}^{2n}$. The function $(u^a/\eta) + x_a$ of (21) is the analog of the harmonic coordinate of refs. 12 and 4. By carrying the term Mx_a to the RHS, and multiplying both sides by η, we rewrite Eq. (21) as

$$\sum_{i=1}^{2n} \Omega_i \left(\partial_{y_i}^2 - \frac{y_i^2}{4} + \frac{1}{2}\right) u^a + \sum_{i=1}^{2n} y_i v_i \nabla_x u^a = -\sum_{i=1}^{2n} y_i v_i^a(x)\, \eta(y) \tag{22}$$

Let us introduce notation needed for the statement of the theorem on the existence and uniqueness of solutions to Eq. (22). $\mathscr{S}^\perp$ is the space of functions on $\mathbb{T}_p^3 \times \mathbb{R}^{2n}$ which are infinitely smooth, orthogonal to η, and decay faster than any polynomial together with all their derivatives. That is $f \in \mathscr{S}^\perp$ if

$$\iint f(x, y)\, \eta(y)\, dx\, dy = 0 \qquad \text{and} \qquad \sup_{x,\, y} Q(y)\, P_1(D_y)\, P_2(D_x)\, f < \infty$$

for any polynomials P_1, P_2, and Q. $\mathscr{L}_2^\perp$ is the completion of $\mathscr{S}^\perp$ in $\mathscr{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n})$. Clearly

$$\mathscr{L}_2^\perp \oplus \{\text{const} \cdot \eta(y)\} = \mathscr{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n})$$

$\|\cdot\|$ is the usual norm of $\mathscr{L}_2(\mathbb{T}_p^3 \times \mathbb{R}^{2n})$. $\mathscr{H}^\perp$ is the completion of $\mathscr{S}^\perp$ in the harmonic oscillator inner product

$$(f, g)_{\mathscr{H}^\perp} = \sum_{i=1}^{2n} \iint ((\partial_{y_i} f)(\partial_{y_i} g) + y_i^2 fg + fg)\, dx\, dy$$

Write

$$f = -\sum_{i=1}^{2n} y_i v_i^a(x)\, \eta(y) \tag{23}$$

$$Lu = \sum_{i=1}^{2n} \Omega_i \left(\partial_{y_i}^2 u - \frac{y_i^2}{4} u + \frac{1}{2} u\right), \qquad Au = \sum_{i=1}^{2n} y_i v_i(x)\, \nabla_x u, \qquad T = L + A$$

We drop the superscript a on the solution of Eq. (22). With the above notation Eq. (22) can be written as

$$Tu = f \tag{24}$$

For the proof of the following theorem we refer the reader to ref. 8.

Theorem 3.1 (ref. 8). Suppose Assumptions A and B hold, and $f \in \mathscr{L}_2^{\perp}$. Then for $m \geqslant m_0$, $n \geqslant n_0(m)$ the equation $Tu = f$ has a unique weak solution $u \in \mathscr{H}^{\perp}$. There is a constant $C(m, n)$, such that

$$\|u\|_{\mathscr{H}^{\perp}} \leqslant C(m, n) \, \|f\| \tag{25}$$

Next we state the theorem which provides the relationship between the effective diffusivity and the solution of Eq. (21).

Theorem 3.2. Suppose Assumptions A and B hold. Then for $m \geqslant m_0$, $n \geqslant n_0(m)$ the effective diffusivity $D^{mn,\, ab}$ is expressed in terms of the solution u^a of Eq. (21) by the formula

$$D^{mn,\, ab} = \frac{1}{2} \sum_{i=1}^{2n} \iint_{\mathbb{T}_p^3 \times \mathbb{R}^{2n}} (u^a v_i^b + u^b v_i^a) \, y_i \eta \, dx \, dy \tag{26}$$

Remark. The proof of Theorem 3.2 is based on application of Ito's formula to $((u^a/\eta) + x_a)(X_t, Y_t)$ and to $((u^b/\eta) + x_b)(X_t, Y_t)$. A similar formula expressing the effective diffusivity in terms of the solution of an elliptic equation was employed in related problems, such as ref. 3. The difference in our case is that the operator in the left hand side of the Eq. (21), which is the infinitesimal generator of the process, is not elliptic, but only hypoelliptic. Therefore, rather that applying the general elliptic theory we need Theorem 3.1 in order to show the existence and uniqueness of the solution. For details see ref. 8.

4. PROOF OF THEOREM 2.2

The proof of Theorem 2.2 is based on the estimate of the solution u^a of Eq. (22). This is a hypoelliptic equation on $2n+3$ dimensional space. The first term on the LHS of (22) is an elliptic operator when considered on R^{2n}. Recall from the Introduction that the coefficients Ω_i are the decay rates of the correlations of the corresponding velocity modes. The bound from below on the coefficients which follows from (11) allows for the estimate of u^a which is independent of m and n.

Proof of Theorem 2.2. We represent the solution u of (22) uniquely as a sum of two functions which are orthogonal in $\mathscr{L}_2(\mathbb{R}^{2n})$ for all x, that is

$$u(x, y) = w(x, y) + u_0(x)\,\eta(y) \tag{27}$$

where $\int u_0(x)\,dx = 0$, and $\int w(x, y)\,\eta(y)\,dy = 0$ as an element of $\mathscr{L}_2(\mathbb{T}_p^3)$. Since $\int \eta^2 y_i\,dy = 0$, the term $u_0(x)\,\eta(y)$ does not contribute to the integral on the RHS of (26). Therefore it is sufficient to estimate the contribution of w to the integral.

Consider the operator T of the LHS of (22) as an (unbounded) operator from $\mathscr{H}^\perp$ to $\mathscr{L}_2^\perp$ with domain $\mathscr{S}^\perp$. We need the following lemma, for the proof of which we refer the reader to ref. 8.

Lemma 4.1 (ref. 8). Under the assumptions of Theorem 2.2, the closure of $T\mathscr{S}^\perp$ coincides with $\mathscr{L}_2^\perp$.

By Lemma 4.1 there exists a sequence $u^k \in \mathscr{S}^\perp$, such that

$$Tu^k = f^k \to f \qquad \text{in } \mathscr{L}_2^\perp \tag{28}$$

Then $\{f^k\}$ is a Cauchy sequence in $\mathscr{L}_2^\perp$, and by (25)

$$u^k \to u \qquad \text{in } \mathscr{H}^\perp \tag{29}$$

Let

$$u^k = w^k + u_0^k \eta$$

$$f^k = g^k + f_0^k \eta$$

as in (27). Note that by (28), (29), and since $\int f\eta\,dy = 0$

$$w^k \to w \qquad \text{in } \mathscr{H}^\perp; \qquad g^k \to f \qquad \text{in } \mathscr{H}^\perp \tag{30}$$

$$u_0^k \to u_0 \qquad \text{in } \mathscr{L}_2(\mathbb{T}_p^3); \qquad f_0^k \to 0 \qquad \text{in } \mathscr{L}_2(\mathbb{T}_p^3) \tag{31}$$

The equation $Tu^k = f^k$ can be written as

$$Lw^k + Au_0^k(x)\,\eta(y) + Aw^k = g^k + f_0^k \eta \tag{32}$$

In order to estimate the contribution of w to the integral in the RHS of (26) we first derive an integral relation satisfied by w^k and u_0^k. In order to do so we multiply (32) successively by η and w^k and integrate in y. We could not perform this integration with w and u_0 replacing w^k and u_0^k in (32), since w may not belong to $\mathscr{S}^\perp$, and thus the integral over y may not converge. Then we let $k \to \infty$ in order to obtain the corresponding relation on w. This relation is used to bound the RHS of (26).

Multiplying (32) by η and integrating in y yields

$$\int \eta L w^k \, dy + \int A u_0^k(x) \, \eta^2(y) \, dy + \int A w^k \eta(y) \, dy = \int g^k \eta \, dy + \int f_0^k \eta^2 \, dy \tag{33}$$

The RHS of (33) is equal to $(1/p^3) \, f_0^k(x)$. Note that also $\int \eta L w^k \, dy = \int w^k L \eta \, dy = 0$ since $L\eta = 0$, and $\int A u_0^k(x) \, \eta^2(y) \, dy = 0$ since $\int y_i \eta^2(y) \, dy = 0$. Thus

$$\int A w^k \eta \, dy = \frac{1}{p^3} f_0^k(x) \tag{34}$$

Multiplying (32) by w^k and integrating in y, we obtain

$$\int w^k L w^k \, dy + \int w^k A u_0^k(x) \, \eta(y) \, dy + \int w^k A w^k \, dy = \int g^k w^k \, dy + \int f_0^k(x) \, \eta w^k \, dy \tag{35}$$

Note that

$$\int w^k A w^k \, dx = \sum_{i=1}^{2n} y_i \int w^k v_i(x) \, \nabla_x w^k \, dx = \tfrac{1}{2} \sum_{i=1}^{2n} y_i \int \operatorname{div}(v_i (w^k)^2) \, dx = 0$$

and thus the last term of the LHS of (35) vanishes after integration over x. The second term on the RHS of (35) vanishes since $\int \eta w^k \, dy = 0$. By (34), since $A^* = -A$

$$\iint w^k A u_0^k(x) \, \eta(y) \, dx \, dy = -\iint u_0^k A w^k \eta(y) \, dx \, dy = -\frac{1}{p^3} \int f_0^k(x) \, u_0^k(x) \, dx$$

Therefore, by (35), we obtain the desired relation on w^k and u_0^k,

$$\iint w^k L w^k \, dx \, dy = \iint g^k w^k \, dx \, dy + \frac{1}{p^3} \int f_0^k(x) \, u_0^k(x) \, dx \tag{36}$$

In order to obtain an integral relation of this type for the limit w we consider $k \to \infty$ in (36). By (30) and (31)

$$\iint w L w \, dx \, dy = \iint f w \, dx \, dy \tag{37}$$

We now use (37) to bound the RHS of (26). Note that $\{p^{3/2}y_i\eta, i=1,..., 2n\}$ is an orthonormal system in $\mathscr{L}_2(\mathbb{R}^{2n})$. Let $a_i(x) = p^{3/2}\int wy_i\eta\, dy$, and let $w_i(x, y) = p^{3/2}a_i(x)\, y_i\eta$. Thus for each x the sum $\sum_{i=1}^{2n} w_i$ is the projection in $\mathscr{L}_2(\mathbb{R}^{2n})$ of w on the subspace spanned by the functions $\{y_i\eta\}$. In particular by (23) the function $\tilde{w} = w - \sum_{i=1}^{2n} w_i$ is orthogonal to f in $\mathscr{L}_2(\mathbb{R}^{2n})$ for each x. Since $L(y_i\eta) = -\Omega_i y_i\eta$, substituting

$$w = \tilde{w} + \sum_{i=1}^{2n} w_i \tag{38}$$

into (37) we obtain

$$\iint \tilde{w}L\tilde{w}\, dx\, dy - \sum_{i=1}^{2n} \Omega_i \int a_i^2(x)\, dx = p^{-3/2} \sum_{i=1}^{n} \int a_i(x)\, v_i^a(x)\, dx \tag{39}$$

Since $\iint \tilde{w}L\tilde{w}\, dx\, dy \leqslant 0$ we conclude from (39) that

$$\sum_{i=1}^{2n} \Omega_i \int a_i^2(x)\, dx \leqslant p^{-3/2} \sum_{i=1}^{2n} \int |a_i(x)\, v_i^a(x)|\, dx \tag{40}$$

By Schwartz inequality, the RHS of (40) is estimated as follows

$$\begin{aligned} &p^{-3/2} \sum_{i=1}^{2n} \int |a_i(x)\, v_i^a(x)|\, dx \\ &\quad\leqslant \frac{c_2}{2} \sum_{i=1}^{2n} (1+|k_i|)^{\beta} \int a_i^2(x)\, dx + \frac{2}{p^3 c_2} \sum_{i=1}^{2n} (1+|k_i|)^{-\beta} \int (v_i^a)^2\,(x)\, dx \end{aligned} \tag{41}$$

where c_2 and β are the same as in (11). Note that by (17) the second term on the RHS of (41) is bounded from above by $(2/c_2) \sum_{i=1}^{n} (1+|k_i|)^{-\beta} (\lambda_i^1 + \lambda_i^2)$, where λ_i^1 and λ_i^2 are the eigenvalues of M_i^{mn}. From (10) and from the definition of the matrices M_i^{mn} it follows that $\lambda_i^1 + \lambda_i^2 \leqslant c(1+|k_i|)^{\alpha} \operatorname{vol}(\Delta_i)$. Thus the sum $\sum_{i=1}^{n} (1+|k_i|)^{-\beta} (\lambda_i^1 + \lambda_i^2)$ is bounded form above by $c \sum_{i=1}^{n} (1+|k_i|)^{-\beta+\alpha} \operatorname{vol}(\Delta_i)$, which is an integral sum for the integral $\int_{|k^a| \leqslant m} (1+|k|)^{\alpha-\beta}\, dk$. Since $\alpha - \beta + 3 < 0$ these integrals are uniformly bounded in m, and therefore the second term on the RHS of (41) is bounded from above by a constant independent of m and n. Therefore by (40), (41), and (11) there exists a constant C such that

$$\sum_{i=1}^{2n} (1+|k_i|)^{\beta} \int a_i^2(x)\, dx \leqslant C \tag{42}$$

In order to bound the RHS of (26) is is sufficient to bound $\sum_{i=1}^{2n} \iint u^a v_i^b y_i \eta \, dx \, dy$. By (27) and (38)

$$\sum_{i=1}^{2n} \iint u^a v_i^b y_i \eta \, dx \, dy = \sum_{i=1}^{2n} \iint w_i v_i^b y_i \eta \, dx \, dy = p^{-3/2} \sum_{i=1}^{2n} \int a_i(x) \, v_i^b \, dx \qquad (43)$$

By Schwartz inequality, the RHS of (43) is bounded from above by

$$\sum_{i=1}^{2n} (1+|k_i|)^{\beta} \int a_i^2(x) \, dx + p^{-3} \sum_{i=1}^{2n} (1+|k_i|)^{-\beta} \int (v_i^b)^2 \, (x) \, dx$$

The first term is bounded from above by C due to (42), and the second term is bounded by another constant, due to (17) and (10) since $\alpha - \beta + 3 < 0$. Therefore the RHS of (26) is bounded from above by a constant. This completes the proof of Theorem 2.2.

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